

Soliton solutions for coupled Schrödinger systems with sign-changing potential

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Abstract In this paper, a class of coupled systems of nonlinear Schrödinger equations with sign-changing potential, including the linearly coupled case, is considered. The existence of non-trivial bound state solutions via linking methods for cones in Banach spaces is proved.

Key words coupled Schrödinger system, sign-changing potential, cohomological index

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1 Introduction and main results

Recently, many mathematicians focused their attention to coupled nonlinear Schrödinger systems. From the viewpoint of physics, coupled Schrödinger systems arise from the models of a lot of natural phenomena. A typical example is the study of the dynamics of coupled Bose-Einstein condensates and the following equation is derived

$$\begin{cases} i\frac{\partial\psi_1}{\partial t} = (-\partial^2/\partial x^2 + V_1 + U_{11}|\psi_1|^2 + U_{12}|\psi_2|^2)\psi_1 + \lambda\psi_2, \\ i\frac{\partial\psi_2}{\partial t} = (-\partial^2/\partial x^2 + V_2 + U_{22}|\psi_2|^2 + U_{21}|\psi_1|^2)\psi_2 + \lambda\psi_1. \end{cases} \quad (1.1)$$

Such systems of equations also appear in nonlinear optical models and many other physical contexts, see [7] for detail discussions. For such coupled systems, the solutions of the form $\psi_j = u_j \exp(i\omega_j t)$ (standing waves) are interesting, where u_j solve the following system

$$\begin{cases} -\frac{\partial^2 u_1}{\partial x^2} + (V_1 + \omega_1)u_1 = -(U_{11}|u_1|^2 + U_{12}|u_2|^2)u_1 - \lambda u_2, \\ -\frac{\partial^2 u_2}{\partial x^2} + (V_2 + \omega_2)u_2 = -(U_{22}|u_2|^2 + U_{21}|u_1|^2)u_2 - \lambda u_1. \end{cases} \quad (1.2)$$

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In this paper, we will consider the following coupled system of nonlinear Schrödinger equations

$$\begin{cases} -\Delta u_1 + (b_1(x) - \lambda V_1(x))u_1 = W_t(x, u_1, u_2) + \lambda \gamma(x)u_2, \\ -\Delta u_2 + (b_2(x) - \lambda V_2(x))u_2 = W_s(x, u_1, u_2) + \lambda \gamma(x)u_1, \\ u_1, u_2 \in H^1(\mathbf{R}^N), \end{cases} \quad (1.3)$$

here and in the sequel, $V_i \in L^\infty(\mathbf{R}^N)$, $\gamma \in L^\infty(\mathbf{R}^N)$, $i = 1, 2$, $\nabla_z W = (W_t, W_s)$ is the gradient of $W(x, t, s)$ with respect to $z = (t, s) \in \mathbf{R}^2$ and we will write $W(x, z) = W(x, t, s)$ for convenience. We divide our discussions into two cases.

The non-radially symmetric case. We assume $b_i(x)$ satisfying the following conditions

- (B) for $i = 1, 2$, $b_i \in C(\mathbf{R}^N)$, there exists a constant $b_i^0 > 0$ such that $\inf_{x \in \mathbf{R}^N} b_i(x) \geq b_i^0$, and the n dimensional Lebesgue measure $meas\{x \in \mathbf{R}^N | b_i(x) \leq M\} < \infty$ for any $M > 0$.

We assume W satisfying the following conditions.

- (W₁) $W \in C^1(\mathbf{R}^N \times \mathbf{R}^2)$, there exists $p \in (2, 2^*)$ such that $0 \leq W(x, z) \leq C(1 + |z|^p)$, $\forall (x, z) \in \mathbf{R}^N \times \mathbf{R}^2$, here, $2^* = \frac{2N}{N-2}$ if $N > 2$ and $2^* = +\infty$ if $N = 1, 2$,
- (W₂) $\lim_{|z| \rightarrow \infty} \frac{W(x, z)}{|z|^2} = +\infty$ uniformly for $x \in \mathbf{R}^N$,
- (W₃) $W_t(x, 0, s) = 0$, $W_s(x, t, 0) = 0$ for any $x \in \mathbf{R}^N$, $s \in \mathbf{R}$, $t \in \mathbf{R}$, and $\lim_{|z| \rightarrow 0} \frac{W(x, z)}{|z|^2} = 0$ uniformly for $x \in \mathbf{R}^N$,
- (W₄) set $\mathcal{W}(x, z) = \nabla_z W(x, z) \cdot z - 2W(x, z)$, then there exists $\theta \geq 1$ such that $\theta \mathcal{W}(x, z) \geq \mathcal{W}(x, \eta z)$, $\forall (x, z) \in \mathbf{R}^N \times \mathbf{R}^2$ and $\eta \in [0, 1]$.

Remark. (1) From (W₄) and $\mathcal{W}(x, 0) = 0$, we see that $\mathcal{W}(x, z) \geq 0$ for any $(x, z) \in \mathbf{R}^N \times \mathbf{R}^2$ by taking $\eta = 0$. So we have $\nabla_z W(x, z) \cdot z \geq 2W(x, z)$.

(2) From condition (W₃), when $\lambda \gamma(x) \neq 0$, $\forall x \in \mathbf{R}^N$, for a non-trivial solution $\mathbf{u} = (u_1, u_2)$ of the problem (1.3), it is easy to see that $u_1 \neq 0$ and $u_2 \neq 0$, so \mathbf{u} does not have an immediate counterpart for a single equation. We also remind that under the above conditions the potential $b_i(x) - \lambda V_i(x)$ may change sign since $\lambda \in \mathbf{R}$, see Theorem 1.1 below.

In this case, we have the following main result.

Theorem 1.1 *If (B) and (W₁)–(W₄) hold, the problem (1.3) possesses a non-trivial solution for every $\lambda \in \mathbf{R}$.*

The radially symmetric case. We assume that $b_i(x)$ satisfy the following condition

(B)_r for $i = 1, 2$, $b_i \in C(\mathbf{R}^N)$, there exists a constant $b_i^0 > 0$ such that $\inf_{x \in \mathbf{R}^N} b_i(x) \geq b_i^0$,
and b_i are radially symmetric, i.e., $b_i(x) = b_i(|x|), \forall x \in \mathbf{R}^N$,

and $V_i(x)$, $\gamma(x)$, $W(x, z)$ further satisfy

(V)_r for $i = 1, 2$, $V_i(x) = V_i(|x|)$, $\gamma(x) = \gamma(|x|), \forall x \in \mathbf{R}^N$.

(W₅) $W(x, z) = W(|x|, z), \forall (x, z) \in \mathbf{R}^N \times \mathbf{R}^2$.

For this case we have the following result.

Theorem 1.2 *If (B)_r, (V)_r and (W₁)–(W₅) hold, the problem (1.3) possesses a non-trivial radially symmetric solution for every $\lambda \in \mathbf{R}$.*

Next, we consider some special cases of (1.3). Firstly, we consider some linearly coupled systems. Precisely, we assume that $W_t(x, t, s)$ dose not depend on s and $W_s(x, t, s)$ does not depend on t , that is to say one can write (1.3) as

$$\begin{cases} -\Delta u_1 + (b_1(x) - \lambda V_1(x))u_1 = f(x, u_1) + \lambda \gamma(x)u_2, \\ -\Delta u_2 + (b_2(x) - \lambda V_2(x))u_2 = g(x, u_2) + \lambda \gamma(x)u_1, \\ u_1, u_2 \in H^1(\mathbf{R}^N). \end{cases} \quad (1.4)$$

In this case, we assume that $f, g \in C(\mathbf{R}^N \times \mathbf{R})$ satisfy

(f₁) $\exists p_1 \in (2, 2^*)$ such that $|f(x, t)| \leq C(1 + |t|^{p_1-1})$, $f(x, t)t \geq 0, \forall (x, t) \in \mathbf{R}^N \times \mathbf{R}$,

(f₂) set $F(x, t) = \int_0^t f(x, t)dt$, $\lim_{|t| \rightarrow \infty} \frac{F(x, t)}{|t|^2} = +\infty$ uniformly in $x \in \mathbf{R}^N$,

(f₃) $\lim_{t \rightarrow 0} \frac{f(x, t)}{t} = 0$ uniformly in $x \in \mathbf{R}^N$,

(f₄) $\mathcal{F}(x, t) = f(x, t)t - 2F(x, t)$, then there exists $\theta_1 \geq 1$ such that $\theta_1 \mathcal{F}(x, t) \geq \mathcal{F}(x, \eta t)$,
 $\forall (x, t) \in \mathbf{R}^N \times \mathbf{R}$ and $\eta \in [0, 1]$,

(g₁) $\exists p_2 \in (2, 2^*)$ such that $|g(x, s)| \leq C(1 + |s|^{p_2-1})$, $g(x, s)s \geq 0, \forall (x, s) \in \mathbf{R}^N \times \mathbf{R}$,

(g₂) set $G(x, s) = \int_0^s g(x, s)ds$, $\lim_{|s| \rightarrow \infty} \frac{G(x, s)}{|s|^2} = +\infty$ uniformly in $x \in \mathbf{R}^N$,

(g₃) $\lim_{s \rightarrow 0} \frac{g(x, s)}{s} = 0$ uniformly in $x \in \mathbf{R}^N$,

(g₄) $\mathcal{G}(x, s) = g(x, s)s - 2G(x, s)$, then there exists $\theta_2 \geq 1$ such that $\theta_2 \mathcal{G}(x, s) \geq \mathcal{G}(x, \eta s)$,
 $\forall (x, s) \in \mathbf{R}^N \times \mathbf{R}$ and $\eta \in [0, 1]$.

Theorem 1.3 *If (B), (f₁)–(f₄) and (g₁)–(g₄) hold, the problem (1.4) possesses a non-trivial solution for every $\lambda \in \mathbf{R}$.*

Proof. Set $W(x, t, s) = F(x, t) + G(x, s)$, it is easy to see that (W₁) and (W₄) hold.

As for (W₂), from (f₂) and (g₂), $\forall M > 0$, there exists $R > 0$ such that $\frac{F(x, t)}{|t|^2} > 2M$ when $|t| \geq R$ and $\frac{G(x, s)}{|s|^2} > 2M$ when $|s| \geq R$. Then

$$\frac{F(x, t) + G(x, s)}{t^2 + s^2} \geq \frac{F(x, t) + G(x, s)}{2 \max(|t|^2, |s|^2)} > M$$

when $\max(|t|, |s|) \geq R$. So $\lim_{|z| \rightarrow \infty} \frac{W(x, z)}{|z|^2} = +\infty$ uniformly for $x \in \mathbf{R}^N$.

From (f₃), (g₃) and the continuity of f and g , we can see $f(x, 0) = 0 = g(x, 0)$, so $W_t(x, 0, s) = 0$, $W_s(x, t, 0) = 0$ for any $x \in \mathbf{R}^N$, $s \in \mathbf{R}$, $t \in \mathbf{R}$. Also from (f₃) and (g₃), we have $\lim_{|t| \rightarrow 0} \frac{F(x, t)}{|t|^2} = 0$ and $\lim_{|s| \rightarrow 0} \frac{G(x, s)}{|s|^2} = 0$, so

$$0 \leq \frac{F(x, t) + G(x, s)}{|t|^2 + |s|^2} \leq \frac{F(x, t)}{|t|^2} + \frac{G(x, s)}{|s|^2} \rightarrow 0.$$

So (W₃) holds. From Theorem 1.1, we get the assertion. ■

As in Theorem 1.2, assuming that $f(x, t)$ and $g(x, s)$ further satisfy

$$(f_5) \quad f(x, t) = f(|x|, t), \text{ for any } (x, t) \in \mathbf{R}^N \times \mathbf{R},$$

$$(g_5) \quad g(x, s) = g(|x|, s), \text{ for any } (x, s) \in \mathbf{R}^N \times \mathbf{R},$$

also setting $W(x, t, s) = F(x, t) + G(x, s)$ and by the same reason as in the proof of Theorem 1.3, we have the following consequence.

Theorem 1.4 *If (B)_r, (V)_r, (f₁)–(f₅) and (g₁)–(g₅) hold, the problem (1.4) possesses a non-trivial radially symmetric solution for every $\lambda \in \mathbf{R}$.*

By taking $f(x, t) = c_1(x)|t|^{p_1-2}t$ and $g(x, s) = c_2(x)|s|^{p_2-2}s$ with $c_i \in L^\infty(\mathbf{R}^N)$ and $\inf_{x \in \mathbf{R}^N} c_i(x) > 0$, $i = 1, 2$, we get the following system

$$\begin{cases} -\Delta u_1 + (b_1(x) - \lambda V_1(x))u_1 = c_1(x)|u_1|^{p_1-2}u_1 + \lambda \gamma(x)u_2, \\ -\Delta u_2 + (b_2(x) - \lambda V_2(x))u_2 = c_2(x)|u_2|^{p_2-2}u_2 + \lambda \gamma(x)u_1, \\ u, v \in H^1(\mathbf{R}^N), \end{cases} \quad (1.5)$$

then for $p_1, p_2 \in (2, 2^*)$, we have the following consequences.

Corollary 1.5 *If (B) holds, the problem (1.5) possesses a non-trivial solution for every $\lambda \in \mathbf{R}$.*

Corollary 1.6 *If $(B)_r$, $(V)_r$ hold, and $c_i(x) = c_i(|x|)$ for any $x \in \mathbf{R}^N$, $i = 1, 2$, the problem (1.5) possesses a non-trivial radially symmetric solution for every $\lambda \in \mathbf{R}$.*

Secondly, by taking $W(x, t, s) = \frac{1}{4}t^4 + \frac{1}{2}t^2s^2 + \frac{1}{4}s^4$, we get the following systems

$$\begin{cases} -\Delta u_1 + (b_1(x) - \lambda V_1(x))u_1 = u_1^3 + u_2^2u_1 + \lambda\gamma(x)u_2, \\ -\Delta u_2 + (b_2(x) - \lambda V_2(x))u_2 = u_2^3 + u_1^2u_2 + \lambda\gamma(x)u_1, \\ u_1, u_2 \in H^1(\mathbf{R}^N). \end{cases} \quad (1.6)$$

as consequences of Theorem 1.1 and 1.2, we have

Corollary 1.7 *If (B) holds, the problem (1.6) possesses a non-trivial solution for every $\lambda \in \mathbf{R}$.*

Corollary 1.8 *If $(B)_r$ and $(V)_r$ hold, the problem (1.6) possesses a non-trivial radially symmetric solution for every $\lambda \in \mathbf{R}$.*

The study of linearly coupled Schrödinger systems from the mathematical point of view began very recently, see [1, 3, 4, 7]. In [3], the authors proved the existence of positive ground state solution of the following system of nonlinear Schrödinger equations for $0 < \lambda < 1$,

$$\begin{cases} -\Delta u + u = (1 + a(x))|u|^{p-2}u + \lambda v, \\ -\Delta v + v = (1 + b(x))|v|^{p-2}v + \lambda u, \\ u, v \in H^1(\mathbf{R}^N), \end{cases} \quad (1.7)$$

with $a, b \in L^\infty(\mathbf{R}^N)$, $\lim_{|x| \rightarrow \infty} a(x) = \lim_{|x| \rightarrow \infty} b(x) = 0$, $\inf_{\mathbf{R}^N} \{1 + a(x)\} > 0$, $\inf_{\mathbf{R}^N} \{1 + b(x)\} > 0$ and $a(x) + b(x) \geq 0$. In [4], the authors devoted to the study the multi-bump solitons of the following system

$$\begin{cases} -\Delta u + u - u^3 = \epsilon v, \\ -\Delta v + v - v^3 = \epsilon u, \\ u, v \in H^1(\mathbf{R}^N), \end{cases} \quad (1.8)$$

in \mathbf{R}^N with dimension $N = 1, 2, 3$. In [1], A. Ambrosetti studied the following two systems

$$\begin{cases} -u_1'' + u_1 = (1 + \varepsilon a_1(x))u_1^3 + \gamma u_2, \\ -u_2'' + u_2 = (1 + \varepsilon a_2(x))u_2^3 + \gamma u_1, \\ u_1, u_2 \in H^1(\mathbf{R}), \end{cases} \quad (1.9)$$

$$\begin{cases} -\varepsilon^2 u_1'' + u_1 + U_1(x)u_1 = u_1^3 + \gamma u_2, \\ -\varepsilon^2 u_2'' + u_2 + U_2(x)u_2 = u_2^3 + \gamma u_1, \\ u_1, u_2 \in H^1(\mathbf{R}), \end{cases} \quad (1.10)$$

and proved the existence of non-trivial solution for (1.9) under the conditions $a_i \in L^\infty(\mathbf{R})$, $\lim_{|x| \rightarrow \infty} a_i(x) = 0$, $i = 1, 2$, $0 < \gamma < 1$, $\gamma \neq 3/5$, and (1.10) possesses a solution concentrating at nondegenerate stationary points of the sum $U_1 + U_2$ when $\varepsilon \rightarrow 0$ under the conditions $U_i \in L^\infty$ and $\inf_{x \in \mathbf{R}} U_i(x) > -1$, $i = 1, 2$. The main tools in [1, 3, 4] are the perturbation techniques, we refer [5] for detailed discussions about these methods. In [7], the following system was considered

$$\begin{cases} -u_1'' + a(x)u_1 - b(x)u_2 = c(x)H_1(u_1, u_2)u_1, \\ -u_2'' + d(x)u_2 - e(x)u_1 = f(x)H_2(u_1, u_2)u_2, \\ u_1, u_2 \in H^1(\mathbf{R}), \end{cases} \quad (1.11)$$

the authors got a non-trivial solution via Krasnoselskii fixed point theory. We note that the potentials in systems (1.7)-(1.11) are positive.

To prove the main theorem, we deal with the existence problem of non-trivial solutions by variational methods. We first study an eigenvalue problem, whose eigenfunctions are solutions of (1.3) but without the nonlinear term, then the non-zero critical point of the functional related to the nonlinear perturbation of this eigenvalue problem is a weak solution of (1.3). To find the critical point, we use a critical point theorem developed by Degiovanni and Lancelotti in [10].

The rest of the paper is organized as follows. The variational setting is contained in section 2. In section 3, we study the eigenvalue problem. We prove that there exists a divergent sequence of eigenvalues which are defined by the cohomological index. We prove Theorem 1.1 and 1.2 in section 4.

2 Variational setting

Let $H_1 := \{u_1 \in H^1(\mathbf{R}^N) \mid \int_{\mathbf{R}^N} b_1(x)u_1^2 dx < \infty\}$, then H_1 is a Hilbert Space with inner product $\langle u_1, v_1 \rangle_1 = \int_{\mathbf{R}^N} (\nabla u_1 \cdot \nabla v_1 + b_1(x)u_1 v_1) dx$ and norm $\|u_1\|_1^2 = \langle u_1, u_1 \rangle_1$. Similarly, let $H_2 := \{u_2 \in H^1(\mathbf{R}^N) \mid \int_{\mathbf{R}^N} b_2(x)u_2^2 dx < \infty\}$, then H_2 is a Hilbert Space with inner product $\langle u_2, v_2 \rangle_2 = \int_{\mathbf{R}^N} (\nabla u_2 \cdot \nabla v_2 + b_2(x)u_2 v_2) dx$ and norm $\|u_2\|_2^2 = \langle u_2, u_2 \rangle_2$.

For the non-radially symmetric case, by the condition (B), H_1 and H_2 can be compactly embedded into $L^p(\mathbf{R}^N)$, $2 \leq p < 2^*$ (see for example, [6, 17]). Set $\mathcal{H} := H_1 \times H_2$, then \mathcal{H} is a Hilbert space with inner product $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 + \langle \cdot, \cdot \rangle_2$ and with norm $\|\mathbf{u}\|^2 = \|u_1\|_1^2 + \|u_2\|_2^2$ for $\mathbf{u} = (u_1, u_2)$.

For the radially symmetric case, let $H_{1,r} := \{u_1 \in H_1 \mid u_1 \text{ is radially symmetric}\}$,

$H_{2,r} := \{u_2 \in H_2 | u_2 \text{ is radially symmetric}\}$, then $H_{i,r}$ is a Hilbert Space with inner product $\langle \cdot, \cdot \rangle_i$ and norm $\| \cdot \|_i$ for $i = 1, 2$. By condition (B)_r, $H_{i,r}$ can be compactly embedded into $L^p(\mathbf{R}^N)$, $2 \leq p < 2^*$ for $i = 1, 2$ (see [6, 17]). In this case, we set $\mathcal{H}_r := H_{1,r} \times H_{2,r}$, then \mathcal{H}_r is a Hilbert space with inner product $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_1 + \langle \cdot, \cdot \rangle_2$ and with norm $\|\mathbf{u}\|^2 = \|u_1\|_1^2 + \|u_2\|_2^2$ for $\mathbf{u} = (u_1, u_2)$.

In order to prove Theorem 1.1, we define a functional $\Psi : \mathcal{H} \rightarrow \mathbf{R}$ by

$$\Psi(\mathbf{u}) = E(\mathbf{u}) - \lambda J(\mathbf{u}) - P(\mathbf{u}), \quad \mathbf{u} = (u_1, u_2) \in \mathcal{H}, \quad (2.12)$$

where

$$E(\mathbf{u}) = \frac{1}{2} \|\mathbf{u}\|^2, \quad (2.13)$$

$$J(\mathbf{u}) = \int_{\mathbf{R}^N} \left(\frac{1}{2} V_1(x) u_1^2 + \gamma(x) u_1 u_2 + \frac{1}{2} V_2(x) u_2^2 \right) dx, \quad (2.14)$$

and

$$P(\mathbf{u}) = \int_{\mathbf{R}^N} W(x, \mathbf{u}) dx = \int_{\mathbf{R}^N} W(x, u_1, u_2) dx, \quad (2.15)$$

then these four functionals are C^1 , and for $\mathbf{u} = (u_1, u_2)$, $\mathbf{v} = (v_1, v_2) \in \mathcal{H}$, there hold

$$\langle E'(\mathbf{u}), \mathbf{v} \rangle = \int_{\mathbf{R}^N} (\nabla u_1 \cdot \nabla v_1 + b_1(x) u_1 v_1) dx + \int_{\mathbf{R}^N} (\nabla u_2 \cdot \nabla v_2 + b_2(x) u_2 v_2) dx, \quad (2.16)$$

$$\langle J'(\mathbf{u}), \mathbf{v} \rangle = \int_{\mathbf{R}^N} (V_1(x) u_1 v_1 + \gamma(x) u_2 v_1 + \gamma(x) u_1 v_2 + V_2(x) u_2 v_2) dx, \quad (2.17)$$

$$\langle P'(\mathbf{u}), \mathbf{v} \rangle = \int_{\mathbf{R}^N} (W_t(x, u_1, u_2) v_1 + W_s(x, u_1, u_2) v_2) dx, \quad (2.18)$$

$$\langle \Psi'(\mathbf{u}), \mathbf{v} \rangle = \langle E'(\mathbf{u}), \mathbf{v} \rangle - \lambda \langle J'(\mathbf{u}), \mathbf{v} \rangle - \langle P'(\mathbf{u}), \mathbf{v} \rangle. \quad (2.19)$$

It is clear that critical points of Ψ are weak solutions of (1.3).

For the radially symmetric case, we can also define these four functionals and (2.16)-(2.19) hold, the only difference is the domain \mathcal{H} of the functional Ψ is replaced by \mathcal{H}_r . And the critical points of the functional Ψ are radially symmetric weak solutions of (1.3).

In order to find a critical point of Ψ , we need the following critical point theorem. It was proved in [10], where the functional was supposed to satisfy the (PS) condition. Recently, in [9], the author extended it to more general case (the functional space is completely regular topological space or metric space). As observed in [15], if the functional space is a real Banach space, according to the proof of Theorem 6.10 in [9], the Cerami condition is sufficient for the compactness of the set of critical points at a fixed level and the first deformation lemma to hold (see [16]). So this critical point theorem still hold under the Cerami condition.

Theorem 2.1 ([10]) *Let \mathcal{H} be a real Banach space and let C_- , C_+ be two symmetric cones in \mathcal{H} such that C_+ is closed in \mathcal{H} , $C_- \cap C_+ = \{0\}$ and*

$$i(C_- \setminus \{0\}) = i(\mathcal{H} \setminus C_+) = m < \infty.$$

Define the following four sets by

$$D_- = \{u \in C_- \mid \|u\| \leq r_-\},$$

$$S_+ = \{u \in C_+ \mid \|u\| = r_+\},$$

$$Q = \{u + te \mid u \in C_-, t \geq 0, \|u + te\| \leq r_-\}, \quad e \in \mathcal{H} \setminus C_-,$$

$$H = \{u + te \mid u \in C_-, t \geq 0, \|u + te\| = r_-\}.$$

Then $(Q, D_- \cup H)$ links S_+ cohomologically in dimension $m + 1$ over \mathbf{Z}_2 . Moreover, suppose $\Psi \in C^1(\mathcal{H}, \mathbf{R})$ satisfying the Cerami condition, and $\sup_{x \in D_- \cup H} \Psi(x) < \inf_{x \in S^+} \Psi(x)$, $\sup_{x \in Q} \Psi(x) < \infty$. Then Ψ has a critical value $d \geq \inf_{x \in S^+} \Psi(x)$.

For convenience, let us recall the definition and some properties of the cohomological index of Fadell-Rabinowitz for a \mathbf{Z}_2 -set, see [11, 12, 16] for details. For simplicity, we only consider the usual \mathbf{Z}_2 -action on a linear space, i.e., $\mathbf{Z}_2 = \{1, -1\}$ and the action is the usual multiplication. In this case, the \mathbf{Z}_2 -set A is a symmetric set with $-A = A$.

Let E be a normed linear space. We denote by $\mathcal{S}(E)$ the set of all symmetric subsets of E which do not contain the origin of E . For $A \in \mathcal{S}(E)$, denote $\bar{A} = A/\mathbf{Z}_2$. Let $\rho : \bar{A} \rightarrow \mathbf{R}P^\infty$ be the classifying map and $\rho^* : H^*(\mathbf{R}P^\infty) = \mathbf{Z}_2[\omega] \rightarrow H^*(\bar{A})$ the induced homomorphism of the cohomology rings. The cohomological index of A , denoted by $i(A)$, is defined by $\sup\{k \geq 1 : \rho^*(\omega^{k-1}) \neq 0\}$. We list some properties of the cohomological index here for further use in this paper. Let $A, B \in \mathcal{S}(E)$, there hold

- (i1) (**monotonicity**) if $h : A \rightarrow B$ is an odd map, then $i(A) \leq i(B)$,
- (i2) (**continuity**) if C is a closed symmetric subset of A , then there exists a closed symmetric neighborhood N of C in A , such that $i(N) = i(C)$, hence the interior of N in A is also a neighborhood of C in A and $i(\text{int}N) = i(C)$,
- (i3) (**neighborhood of zero**) if V is bounded closed symmetric neighborhood of the origin in E , then $i(\partial V) = \dim E$.

3 The eigenvalue problem

First we solve the eigenvalue problem

$$E'(\mathbf{u}) = \mu J'(\mathbf{u}), \quad \mathbf{u} \in \mathcal{H}. \quad (3.20)$$

Lemma 3.1 *For any $\mathbf{u} = (u_1, u_2), \mathbf{v} = (v_1, v_2) \in \mathcal{H}$, it holds that*

$$\langle E'(\mathbf{u}) - E'(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle \geq (\|u_1\|_1 - \|v_1\|_1)^2 + (\|u_2\|_2 - \|v_2\|_2)^2. \quad (3.21)$$

Proof. By direct computations, we have

$$\begin{aligned} & \langle E'(\mathbf{u}) - E'(\mathbf{v}), \mathbf{u} - \mathbf{v} \rangle \\ &= \int_{\mathbf{R}^N} (|\nabla u_1|^2 + |\nabla v_1|^2 - 2\nabla u_1 \cdot \nabla v_1) \, dx + \int_{\mathbf{R}^N} b_1(x) (|u_1|^2 + |v_1|^2 - 2u_1 v_1) \, dx \\ &+ \int_{\mathbf{R}^N} (|\nabla u_2|^2 + |\nabla v_2|^2 - 2\nabla u_2 \cdot \nabla v_2) \, dx + \int_{\mathbf{R}^N} b_2(x) (|u_2|^2 + |v_2|^2 - 2u_2 v_2) \, dx. \end{aligned}$$

From the definition of the norm in H_i , we can get

$$\begin{aligned} & \int_{\mathbf{R}^N} (|\nabla u_1|^2 + |\nabla v_1|^2 - 2\nabla u_1 \cdot \nabla v_1) \, dx + \int_{\mathbf{R}^N} b_1(x) (|u_1|^2 + |v_1|^2 - 2u_1 v_1) \, dx \\ &= \|u_1\|_1^2 + \|v_1\|_1^2 - 2\langle u_1, v_1 \rangle_1 \geq \|u_1\|_1^2 + \|v_1\|_1^2 - 2\|u_1\|_1 \|v_1\|_1 = (\|u_1\|_1 - \|v_1\|_1)^2, \end{aligned} \quad (3.22)$$

$$\begin{aligned} & \int_{\mathbf{R}^N} (|\nabla u_2|^2 + |\nabla v_2|^2 - 2\nabla u_2 \cdot \nabla v_2) \, dx + \int_{\mathbf{R}^N} b_2(x) (|u_2|^2 + |v_2|^2 - 2u_2 v_2) \, dx \\ &= \|u_2\|_2^2 + \|v_2\|_2^2 - 2\langle u_2, v_2 \rangle_2 \geq \|u_2\|_2^2 + \|v_2\|_2^2 - 2\|u_2\|_2 \|v_2\|_2 = (\|u_2\|_2 - \|v_2\|_2)^2. \end{aligned} \quad (3.23)$$

Now (3.22) and (3.23) imply (3.21). ■

Lemma 3.2 *If $\mathbf{u}_n \rightharpoonup \mathbf{u}$ and $\langle E'(\mathbf{u}_n), \mathbf{u}_n - \mathbf{u} \rangle \rightarrow 0$, then $\mathbf{u}_n \rightarrow \mathbf{u}$ in \mathcal{H} .*

Proof. Since \mathcal{H} is a Hilbert space and $\mathbf{u}_n = (u_n, v_n) \rightharpoonup \mathbf{u} = (u, v)$, we only need to show that $\|\mathbf{u}_n\| \rightarrow \|\mathbf{u}\|$. Note that

$$\lim_{n \rightarrow \infty} \langle E'(\mathbf{u}_n) - E'(\mathbf{u}), \mathbf{u}_n - \mathbf{u} \rangle = \lim_{n \rightarrow \infty} (\langle E'(\mathbf{u}_n), \mathbf{u}_n - \mathbf{u} \rangle - \langle E'(\mathbf{u}), \mathbf{u}_n - \mathbf{u} \rangle) = 0.$$

By inequality (3.21) we have

$$\langle E'(\mathbf{u}_n) - E'(\mathbf{u}), \mathbf{u}_n - \mathbf{u} \rangle \geq (\|u_n\|_1 - \|u\|_1)^2 + (\|v_n\|_2 - \|v\|_2)^2.$$

So $\|u_n\|_1 \rightarrow \|u\|_1$, $\|v_n\|_2 \rightarrow \|v\|_2$ and hence $\|\mathbf{u}_n\| \rightarrow \|\mathbf{u}\|$ as $n \rightarrow \infty$ and the assertion follows. ■

Lemma 3.3 *J' is weak-to-strong continuous, i.e. $\mathbf{u}_n \rightharpoonup \mathbf{u}$ in \mathcal{H} implies $J'(\mathbf{u}_n) \rightarrow J'(\mathbf{u})$.*

Proof. Since $\mathbf{u}_n = (u_n, v_n) \rightharpoonup \mathbf{u} = (u, v)$ in \mathcal{H} , $u_n \rightharpoonup u$ in H_1 . So $u_n \rightarrow u$ in $L^2(\mathbf{R}^N)$ because H_1 compactly embedded into $L^2(\mathbf{R}^N)$. Similarly, we have $v_n \rightarrow v$ in $L^2(\mathbf{R}^N)$.

For any $\mathbf{v} = (\tilde{u}, \tilde{v}) \in \mathcal{H}$,

$$\int_{\mathbf{R}^N} \tilde{u}^2 dx \leq \frac{1}{b_1^0} \int_{\mathbf{R}^N} b_1(x) \tilde{u}^2 dx \leq \frac{1}{b_1^0} \|\tilde{u}\|_1^2 \leq \frac{1}{b_1^0} \|\mathbf{v}\|^2,$$

so $(\int_{\mathbf{R}^N} \tilde{u}^2 dx)^{\frac{1}{2}} \leq C \|\mathbf{v}\|$. Similarly, we have $(\int_{\mathbf{R}^N} \tilde{v}^2 dx)^{\frac{1}{2}} \leq C \|\mathbf{v}\|$. Then,

$$\begin{aligned} & |\langle J'(\mathbf{u}_n) - J'(\mathbf{u}), \mathbf{v} \rangle| \\ &= \left| \int_{\mathbf{R}^N} (V_1(x)(u_n - u)\tilde{u} + \gamma(x)(v_n - v)\tilde{u} + \gamma(x)(u_n - u)\tilde{v} + V_2(x)(v_n - v)\tilde{v}) dx \right| \\ &\leq \|V_1\|_\infty \left(\int_{\mathbf{R}^N} (u_n - u)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^N} \tilde{u}^2 dx \right)^{\frac{1}{2}} + \|\gamma\|_\infty \left(\int_{\mathbf{R}^N} (v_n - v)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^N} \tilde{u}^2 dx \right)^{\frac{1}{2}} \\ &\quad + \|\gamma\|_\infty \left(\int_{\mathbf{R}^N} (u_n - u)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^N} \tilde{v}^2 dx \right)^{\frac{1}{2}} + \|V_2\|_\infty \left(\int_{\mathbf{R}^N} (v_n - v)^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^N} \tilde{v}^2 dx \right)^{\frac{1}{2}} \\ &\leq C \left(\int_{\mathbf{R}^N} (u_n - u)^2 dx \right)^{\frac{1}{2}} \|\mathbf{v}\| + C \left(\int_{\mathbf{R}^N} (v_n - v)^2 dx \right)^{\frac{1}{2}} \|\mathbf{v}\| \rightarrow 0, \end{aligned}$$

hence $J'(\mathbf{u}_n) \rightarrow J'(\mathbf{u})$. ■

Lemma 3.4 *If $\mathbf{u}_n \rightharpoonup \mathbf{u}$ in \mathcal{H} , then $J(\mathbf{u}_n) \rightarrow J(\mathbf{u})$.*

Proof.

$$\begin{aligned} 2|J(\mathbf{u}_n) - J(\mathbf{u})| &= |\langle J'(\mathbf{u}_n), \mathbf{u}_n \rangle - \langle J'(\mathbf{u}), \mathbf{u} \rangle| \\ &= |\langle J'(\mathbf{u}_n) - J'(\mathbf{u}), \mathbf{u}_n \rangle + \langle J'(\mathbf{u}), \mathbf{u}_n - \mathbf{u} \rangle| \\ &\leq \|J'(\mathbf{u}_n) - J'(\mathbf{u})\| \|\mathbf{u}_n\| + o(1). \end{aligned}$$

Because $\mathbf{u}_n \rightharpoonup \mathbf{u}$, \mathbf{u}_n is bounded. From Lemma 3.3, we have $J(\mathbf{u}_n) \rightarrow J(\mathbf{u})$. ■

In this section, we assume that V_1 and V_2 satisfy the following condition

$$(**) \quad \text{meas}\{x \in \mathbf{R}^N \mid V_1(x) > 0\} > 0 \quad \text{or} \quad \text{meas}\{x \in \mathbf{R}^N \mid V_2(x) > 0\} > 0.$$

Set $\mathcal{M} = \{\mathbf{u} \in \mathcal{H} \mid J(\mathbf{u}) = 1\}$, by (**), we can see that \mathcal{M} is not empty, see also Lemma 3.7 below. Clearly, $J(\mathbf{u}) = \frac{1}{2} \langle J'(\mathbf{u}), \mathbf{u} \rangle$, so 1 is a regular value of the functional J . Hence by the implicit theorem, \mathcal{M} is a C^1 -Finsler manifold. It is complete, symmetric, since J is continuous and even. Moreover, 0 is not contained in \mathcal{M} , so the trivial \mathbf{Z}_2 -action on \mathcal{M} is free. Set $\tilde{E} = E|_{\mathcal{M}}$.

Lemma 3.5 *If $\mathbf{u} \in \mathcal{M}$ satisfies $\tilde{E}(\mathbf{u}) = \mu$ and $\tilde{E}'(\mathbf{u}) = 0$, then (μ, \mathbf{u}) is a solution of the functional equation (3.20).*

Proof. By Proposition 3.54 in [16], the norm of $\tilde{E}'(\mathbf{u}) \in T_{\mathbf{u}}^* \mathcal{M}$ is given by $\|\tilde{E}'(\mathbf{u})\|_{\mathbf{u}}^* = \min_{\nu \in \mathbf{R}} \|E'(\mathbf{u}) - \nu J'(\mathbf{u})\|^*$ (here the norm $\|\cdot\|_{\mathbf{u}}^*$ is the norm in the fibre $T_{\mathbf{u}}^* \mathcal{M}$, and $\|\cdot\|^*$ is the operator norm, the minimal can be attained was proved in Lemma 3.55 in [16]). Hence there exists $\nu \in \mathbf{R}$ such that $E'(\mathbf{u}) - \nu J'(\mathbf{u}) = 0$, that is (ν, \mathbf{u}) is a solution of the equation (3.20) and $\mu = \tilde{E}(\mathbf{u}) = \frac{1}{2} \langle E'(\mathbf{u}), \mathbf{u} \rangle = \frac{1}{2} \langle \nu J'(\mathbf{u}), \mathbf{u} \rangle = \frac{\nu}{2} \langle J'(\mathbf{u}), \mathbf{u} \rangle = \nu J(\mathbf{u}) = \nu$. \blacksquare

Lemma 3.6 *\tilde{E} satisfies the (PS) condition, i.e. if (\mathbf{u}_k) is a sequence on \mathcal{M} such that $\tilde{E}(\mathbf{u}_k) \rightarrow c$, and $\tilde{E}'(\mathbf{u}_k) \rightarrow 0$, then up to a subsequence $\mathbf{u}_k \rightarrow \mathbf{u} \in \mathcal{M}$ in \mathcal{H} .*

Proof. First, from the definition of E , we can deduce that (\mathbf{u}_k) is bounded. Then, up to a subsequence, \mathbf{u}_k converges weakly to some \mathbf{u} , by Lemma 3.4, we have $J(\mathbf{u}) = 1$, so $\mathbf{u} \in \mathcal{M}$.

From $\tilde{E}'(\mathbf{u}_k) \rightarrow 0$, we have $E'(\mathbf{u}_k) - \nu_k J'(\mathbf{u}_k) \rightarrow 0$ in \mathcal{H} for a sequence of real numbers (ν_k) . So $\langle E'(\mathbf{u}_k) - \nu_k J'(\mathbf{u}_k), \mathbf{u}_k \rangle \rightarrow 0$, thus we get $\nu_k \rightarrow c$. By Lemma 3.3, we have $E'(\mathbf{u}_k) \rightarrow cJ'(\mathbf{u})$. Hence

$$\langle E'(\mathbf{u}_k), \mathbf{u}_k - \mathbf{u} \rangle = \langle E'(\mathbf{u}_k) - cJ'(\mathbf{u}), \mathbf{u}_k - \mathbf{u} \rangle + \langle cJ'(\mathbf{u}), \mathbf{u}_k - \mathbf{u} \rangle \rightarrow 0.$$

By Lemma 3.2, we obtain $\mathbf{u}_k \rightarrow \mathbf{u}$. \blacksquare

Let \mathcal{F} denote the class of symmetric subsets of \mathcal{M} , $\mathcal{F}_n = \{M \in \mathcal{F} \mid i(M) \geq n\}$ and

$$\mu_n = \inf_{M \in \mathcal{F}_n} \sup_{\mathbf{u} \in M} E(\mathbf{u}). \quad (3.24)$$

Since $\mathcal{F}_n \supset \mathcal{F}_{n+1}$, $\mu_n \leq \mu_{n+1}$.

Lemma 3.7 *If $(**)$ holds, then for every \mathcal{F}_n , there is a compact symmetric set $M \in \mathcal{F}_n$.*

Proof. We follow the idea of the proof of Theorem 3.2 in [13]. Suppose $\text{meas}\{x \in \mathbf{R}^N \mid V_1(x) > 0\} > 0$, it implies that $\forall n \in \mathbb{N}$, there exist n open balls $(B_i)_{1 \leq i \leq n}$ in \mathbf{R}^N such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and $\text{meas}(\{x \in \mathbf{R}^N \mid V_1(x) > 0\} \cap B_i) > 0$. Approximating the characteristic function χ_i of set $\{x \in \mathbf{R}^N \mid V_1(x) > 0\} \cap B_i$ by a C^∞ -function u_i in $L^2(\mathbf{R}^N)$, and require that the sequence $\{u_i\}_{1 \leq i \leq n} \subseteq C^\infty(\mathbf{R}^N)$ satisfies $\int_{\mathbf{R}^N} V_1(x) |u_i|^2 dx > 0$ for all $i = 1, \dots, n$ and $\text{supp } u_i \cap \text{supp } u_j = \emptyset$ when $i \neq j$. Set $\mathbf{u}_i = (u_i, 0) \in \mathcal{H}$, then $J(\mathbf{u}_i) = \frac{1}{2} \int_{\mathbf{R}^N} V_1(x) |u_i|^2 dx > 0$. Normalizing \mathbf{u}_i , we assume that $J(\mathbf{u}_i) = 1$. Denote by U_n the space spanned by $(\mathbf{u}_i)_{1 \leq i \leq n}$. $\forall \mathbf{u} \in U_n$, we have $\mathbf{u} = \sum_{i=1}^n \alpha_i \mathbf{u}_i$ and $J(\mathbf{u}) = \sum_{i=1}^n |\alpha_i|^2$. So $(J(\mathbf{u}))^{\frac{1}{2}}$ defines a norm on U_n . Since U_n is n dimensional, this norm is equivalent to $\|\cdot\|$. Thus $\{\mathbf{u} \in U_n \mid J(\mathbf{u}) = 1\} \subseteq \mathcal{M}$ is compact with respect to the norm $\|\cdot\|$ and by

the property (i3) of cohomological index, $i(\{\mathbf{u} \in U_n \mid J(\mathbf{u}) = 1\}) = n$. So $\{\mathbf{u} \in U_n \mid J(\mathbf{u}) = 1\} \in \mathcal{F}_n$. If $\text{meas}\{x \in \mathbf{R}^N \mid V_2(x) > 0\} > 0$, the proof is similar. \blacksquare

By Lemma 3.7, we have $\mu_n < +\infty$, and by condition (B), there holds $\mu_n \geq 0$. Furthermore, by Lemma 3.6 and Proposition 3.52 in [16], we see that μ_n is sequence of critical values of \tilde{E} and $\mu_n \rightarrow +\infty$, as $n \rightarrow \infty$. By Lemma 3.5 we get a divergent sequence of eigenvalues for problem (3.20). So we have the following result.

Theorem 3.8 *Under the condition (**), the problem (3.20) has an increasing sequence eigenvalues μ_n which are defined by (3.24) and $\mu_n \rightarrow +\infty$, as $n \rightarrow \infty$.*

Lemma 3.9 *Under the condition (**), Set*

$$\rho_n = \inf_{K \in \mathcal{F}_n^c} \sup_{\mathbf{u} \in K} E(\mathbf{u}), \quad (3.25)$$

where $\mathcal{F}_n^c = \{K \in \mathcal{F}_n \mid K \text{ is compact}\}$. we have $\mu_n = \rho_n$.

Proof. From Lemma 3.7, $\mathcal{F}_n^c \neq \emptyset$ and so $\rho_n < +\infty$. It is obvious that $\mu_n \leq \rho_n$. If $\mu_n < \rho_n$, there is $M \in \mathcal{F}_n$ such that $\sup_{\mathbf{u} \in M} E(\mathbf{u}) < \rho_n$. The closure \overline{M} of M in \mathcal{M} is still in \mathcal{F}_n , by continuity of E , $\sup_{\mathbf{u} \in \overline{M}} E(\mathbf{u}) < \rho_n$ holds. By the property (i2) of the cohomological index, we can find a small open neighborhood $A \in \mathcal{F}_n$ of \overline{M} in \mathcal{M} such that $\sup_{\mathbf{u} \in A} E(\mathbf{u}) < \rho_n$. As it was proved in the proof of Proposition 3.1 in [10], for every symmetric open subset A of \mathcal{M} , there holds $i(A) = \sup\{i(K) \mid K \text{ is compact and symmetric with } K \subseteq A\}$. So we can choose a symmetric compact subset $K \subseteq A$ with $i(K) \geq n$ and $\sup_{\mathbf{u} \in K} E(\mathbf{u}) < \rho_n$. This contradicts to the definition of ρ_n . Therefore we have $\mu_n = \rho_n$. \blacksquare

Set $C_m = \{\mathbf{u} \in \mathcal{H} \setminus \{0\} \mid E(\mathbf{u}) \leq \mu_m J(\mathbf{u})\}$ and $D_m = \{\mathbf{u} \in \mathcal{H} \mid E(\mathbf{u}) < \mu_{m+1} J(\mathbf{u})\}$. It is clear that $C_m, D_m \in \mathcal{S}(\mathcal{H})$, i.e., C_m and D_m are symmetric subsets of \mathcal{H} and do not contain 0.

Theorem 3.10 *If $\mu_m < \mu_{m+1}$ for some $m \in \mathbb{N}$, then the cohomological indices satisfy*

$$i(C_m) = i(D_m) = m. \quad (3.26)$$

Proof. Follow the idea of the proof of Theorem 3.2 in [10]. Suppose $\mu_m < \mu_{m+1}$. If we set $A_m = \{\mathbf{u} \in \mathcal{M} \mid E(\mathbf{u}) \leq \mu_m\}$ and $B_m = \{\mathbf{u} \in \mathcal{M} \mid E(\mathbf{u}) < \mu_{m+1}\}$, by the definition (3.24), we have $i(A_m) \leq m$. Assume that $i(A_m) \leq m - 1$. Then, by the property (i2) of the cohomological index, there exists a symmetric neighborhood N of A_m in \mathcal{M} satisfying $i(N) = i(A_m)$. By the equivariant deformation theorem (see [8]), there exists $\delta > 0$ and an odd continuous map $\iota : \{\mathbf{u} \in \mathcal{M} \mid E(\mathbf{u}) \leq \mu_m + \delta\} \rightarrow \{\mathbf{u} \in \mathcal{M} \mid E(\mathbf{u}) \leq \mu_m - \delta\} \cup N = N$.

Hence $i(\mathbf{u} \in \mathcal{M} | E(\mathbf{u}) \leq \mu_m + \delta) \leq m - 1$. By (3.24), there exists $M \in \mathcal{F}_m$ such that $\sup_{\mathbf{u} \in M} E(\mathbf{u}) < \mu_m + \delta$. So $M \subseteq \{\mathbf{u} \in \mathcal{M} | E(\mathbf{u}) \leq \mu_m + \delta\}$ and thus $i(M) \leq m - 1$. This contradicts to the fact that $M \in \mathcal{F}_m$. Thus we have $i(A_m) = m$. By 2-homogeneousness of the functionals E, J , the map $h : C_m \rightarrow A_m$ with $h(\mathbf{u}) = \frac{1}{\sqrt{J(\mathbf{u})}}\mathbf{u}$ is odd, from the monotonicity (i1) of the cohomological index, we have $i(C_m) \leq m$. But it is clear that $A_m \subset C_m$, we have $i(C_m) \geq m$, so $i(C_m) = m$.

Since $A_m \subseteq B_m$ and $i(A_m) = m$, we have $i(B_m) \geq m$. Assume that $i(B_m) \geq m + 1$. As in the proof of Lemma 3.9, there exists a symmetric, compact subset K of B_m with $i(K) \geq m + 1$. Since $\max_{\mathbf{u} \in K} E(\mathbf{u}) < \mu_{m+1}$, this contradicts to definition (3.24). So $i(B_m) = m$. Similar to the above arguments, we also have $i(D_m) = m$. \blacksquare

Remark 3.11 If we consider the following eigenvalue problem,

$$E'(\mathbf{u}) = \mu J'(\mathbf{u}), \quad \mathbf{u} \in \mathcal{H}_r, \quad (3.27)$$

then all the results in this section still hold, we only need to replace the space \mathcal{H} by \mathcal{H}_r .

4 Proof of the main theorems

Replacing (λ, V_i, γ) with $(-\lambda, -V_i, -\gamma)$ if necessary, we can assume that $\lambda \geq 0$. First, we consider the case that condition $(**)$ holds and there exists $m \geq 1$ such that $\mu_m \leq \lambda < \mu_{m+1}$. Set

$$C_- = \{\mathbf{u} \in \mathcal{H} | E(\mathbf{u}) \leq \mu_m J(\mathbf{u})\}, \quad (4.28)$$

$$C_+ = \{\mathbf{u} \in \mathcal{H} | E(\mathbf{u}) \geq \mu_{m+1} J(\mathbf{u})\}. \quad (4.29)$$

It is easy to see that C_-, C_+ are two symmetric closed cones in \mathcal{H} and $C_- \cap C_+ = \{0\}$.

By (3.26) we have

$$i(C_- \setminus \{0\}) = i(C_m) = i(D_m) = i(\mathcal{H} \setminus C_+) = m. \quad (4.30)$$

Lemma 4.1 *There exist $r_+ > 0$ and $\alpha > 0$ such that $\Psi(\mathbf{u}) > \alpha$ for $\mathbf{u} \in C_+$ and $\|\mathbf{u}\| = r_+$.*

Proof. Let $\varepsilon > 0$ be small enough, from (W_1) and (W_3) , we have $|W(x, z)| \leq \varepsilon |z|^2 + C_\varepsilon |z|^p$.

By the Sobolev embedding inequality, for $\mathbf{u} = (u_1, u_2) \in C_+$, we can get

$$\begin{aligned}
\Psi(\mathbf{u}) &= E(\mathbf{u}) - \lambda J(\mathbf{u}) - P(\mathbf{u}) \\
&= E(\mathbf{u}) - \frac{\lambda}{\mu_{m+1}} \cdot \mu_{m+1} J(\mathbf{u}) - P(\mathbf{u}) \\
&\geq E(\mathbf{u}) - \frac{\lambda}{\mu_{m+1}} E(\mathbf{u}) - \varepsilon \int_{\mathbf{R}^N} |u_1|^2 dx \\
&\quad - \varepsilon \int_{\mathbf{R}^N} |u_2|^2 dx - C_\varepsilon \int_{\mathbf{R}^N} |u_1|^p dx - C_\varepsilon \int_{\mathbf{R}^N} |u_2|^p dx \\
&\geq E(\mathbf{u}) - \frac{\lambda}{\mu_{m+1}} E(\mathbf{u}) - \frac{\varepsilon}{b_1^0} \int_{\mathbf{R}^N} b_1(x) |u_1|^2 dx - \frac{\varepsilon}{b_2^0} \int_{\mathbf{R}^N} b_2(x) |u_2|^2 dx \\
&\quad - C_\varepsilon \int_{\mathbf{R}^N} |u_1|^p dx - C_\varepsilon \int_{\mathbf{R}^N} |u_2|^p dx \\
&\geq (1 - \frac{\lambda}{\mu_{m+1}} - 2 \max(\frac{\varepsilon}{b_1^0}, \frac{\varepsilon}{b_2^0})) E(\mathbf{u}) - C_\varepsilon \int_{\mathbf{R}^N} |u_1|^p dx - C_\varepsilon \int_{\mathbf{R}^N} |u_2|^p dx \\
&\geq \frac{1}{2} (1 - \frac{\lambda}{\mu_{m+1}} - 2 \max(\frac{\varepsilon}{b_1^0}, \frac{\varepsilon}{b_2^0})) \|\mathbf{u}\|^2 - C \|\mathbf{u}\|^p.
\end{aligned} \tag{4.31}$$

We remind that in the second inequality of (4.31), the condition (B) has been applied. Since $p > 2$, the assertion follows. \blacksquare

Since $\lambda \geq \mu_m$, by (W₁) it holds that

$$\Psi(\mathbf{u}) \leq 0, \quad \forall \mathbf{u} \in C_-. \tag{4.32}$$

Set $\mathbf{R}^+ = [0, +\infty)$. Following the idea of the proof of Theorem 4.1 in [10], we have

Lemma 4.2 *Let $\mathbf{e} = (e_1, e_2) \in \mathcal{H} \setminus C_-$, there exists $r_- > r_+$ such that $\Psi(\mathbf{u}) \leq 0$ for $\mathbf{u} \in C_- + \mathbf{R}^+ \mathbf{e}$ and $\|\mathbf{u}\| \geq r_-$.*

Proof. Define another norm on \mathcal{H} by $\|\mathbf{u}\|_V^2 := \int_{\mathbf{R}^N} (|V_1(x)| + |\gamma(x)| + 1) |u|^2 dx + \int_{\mathbf{R}^N} (|V_2(x)| + |\gamma(x)| + 1) |v|^2 dx$ for $\mathbf{u} = (u, v)$. Then the same reason as the proof of Theorem 4.1 in [10], there exists some constant $b > 0$ such that $\|\mathbf{u} + t\mathbf{e}\| \leq b \|\mathbf{u} + t\mathbf{e}\|_V$ for every $\mathbf{u} \in C_-$, $t \geq 0$ and some $b > 0$. That is

$$\begin{aligned}
&\int_{\mathbf{R}^N} (|\nabla(u + te_1)|^2 + b_1(x) |u + te_1|^2) dx + \int_{\mathbf{R}^N} (|\nabla(v + te_2)|^2 + b_2(x) |v + te_2|^2) dx \\
&\leq b^2 \int_{\mathbf{R}^N} (|V_1(x)| + |\gamma(x)| + 1) |u + te_1|^2 dx + b^2 \int_{\mathbf{R}^N} (|V_2(x)| + |\gamma(x)| + 1) |v + te_2|^2 dx.
\end{aligned} \tag{4.33}$$

Let $\{\mathbf{u}_k\}$ be a sequence such that $\|\mathbf{u}_k\| \rightarrow +\infty$ and $\mathbf{u}_k \in C_- + \mathbf{R}^+ \mathbf{e}$. Set $\mathbf{v}_k = (u_k, v_k) := \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}$, then, up to a subsequence, $\{\mathbf{v}_k\}$ converges to some $\mathbf{v} = (u_0, v_0)$ weakly in \mathcal{H} and $u_k \rightarrow u_0$, $v_k \rightarrow v_0$ a.e. in \mathbf{R}^N . Note that Lemma 3.4 is also true for functional $\int_{\mathbf{R}^N} (|V_1(x)| + |\gamma(x)| + 1) |u|^2 dx + \int_{\mathbf{R}^N} (|V_2(x)| + |\gamma(x)| + 1) |v|^2 dx$, $\mathbf{u} = (u, v) \in \mathcal{H}$, it follows from (4.33) that $\int_{\mathbf{R}^N} (|V_1(x)| + |\gamma(x)| + 1) |u_0|^2 dx + \int_{\mathbf{R}^N} (|V_2(x)| + |\gamma(x)| + 1) |v_0|^2 dx \geq \frac{1}{b^2}$. So $|\mathbf{v}| \neq 0$ on a positive measure set Ω_0 . By (W₂) we have

$$\lim_{k \rightarrow \infty} \frac{W(x, \mathbf{u}_k(x))}{\|\mathbf{u}_k\|^2} = \lim_{k \rightarrow \infty} \frac{W(x, \|\mathbf{u}_k\| \mathbf{v}_k(x))}{\|\mathbf{u}_k\|^2 |\mathbf{v}_k(x)|^2} |\mathbf{v}_k(x)|^2 = +\infty, \quad x \in \Omega_0.$$

By (W₁) and Fatou lemma we can get

$$\frac{\int_{\mathbf{R}^N} W(x, \mathbf{u}_k(x)) dx}{\|\mathbf{u}_k\|^2} \rightarrow +\infty, \text{ as } k \rightarrow \infty.$$

By the arbitrariness of the sequence $\{\mathbf{u}_k\}$, we have

$$\frac{\int_{\mathbf{R}^N} W(x, \mathbf{u}(x)) dx}{\|\mathbf{u}\|^2} \rightarrow +\infty \quad (4.34)$$

as $\|\mathbf{u}\| \rightarrow +\infty$ and $\mathbf{u} \in C_- + \mathbf{R}^+ \mathbf{e}$. Noting that

$$\frac{\Psi(\mathbf{u})}{\|\mathbf{u}\|^2} = \frac{1}{2} - \frac{\lambda J(\mathbf{u})}{\|\mathbf{u}\|^2} - \frac{\int_{\mathbf{R}^N} W(x, \mathbf{u}(x)) dx}{\|\mathbf{u}\|^2} \quad (4.35)$$

and by conditions (B) and (V), for $\mathbf{u} = (u, v) \in \mathcal{H}$

$$\left| \frac{J(\mathbf{u})}{\|\mathbf{u}\|^2} \right| \leq \frac{C(\int_{\mathbf{R}^N} |u|^2 dx + \int_{\mathbf{R}^N} |v|^2 dx)}{\|\mathbf{u}\|^2} \leq \frac{C(\int_{\mathbf{R}^N} b_1(x) |u|^2 dx + \int_{\mathbf{R}^N} b_2(x) |v|^2 dx)}{\|\mathbf{u}\|^2} \leq C, \quad (4.36)$$

the assertion follows from (4.34), (4.35) and (4.36). \blacksquare

Lemma 4.3 *Ψ satisfies the Cerami condition, i.e., for any sequence $\{\mathbf{u}_k\}$ in \mathcal{H} satisfying $(1 + \|\mathbf{u}_k\|)\Psi'(\mathbf{u}_k) \rightarrow 0$ and $\Psi(\mathbf{u}_k) \rightarrow c$ possesses a convergent subsequence.*

Proof. Let $\{\mathbf{u}_k\}$ be a sequence in \mathcal{H} satisfying $(1 + \|\mathbf{u}_k\|)\Psi'(\mathbf{u}_k) \rightarrow 0$ and $\Psi(\mathbf{u}_k) \rightarrow c$.

We claim that $\{\mathbf{u}_k\}$ is bounded in \mathcal{H} . Otherwise, if $\|\mathbf{u}_k\| \rightarrow \infty$, we consider $\mathbf{v}_k := \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}$.

Then, up to subsequence, we get $\mathbf{v}_k \rightharpoonup \mathbf{v}$ in \mathcal{H} and $\mathbf{v}_k \rightarrow \mathbf{v}$ a.e. in \mathbf{R}^N .

If $\mathbf{v} \neq 0$ in \mathcal{H} , since $\Psi'(\mathbf{u}_k)\mathbf{u}_k \rightarrow 0$, that is to say

$$\begin{aligned} & \|\mathbf{u}_k\|^2 - \lambda J'(\mathbf{u}_k) \cdot \mathbf{u}_k - \int_{\mathbf{R}^N} \nabla_z W(x, \mathbf{u}_k(x)) \cdot \mathbf{u}_k dx \\ &= \|\mathbf{u}_k\|^2 - 2\lambda J(\mathbf{u}_k) - \int_{\mathbf{R}^N} \nabla_z W(x, \mathbf{u}_k(x)) \cdot \mathbf{u}_k dx \rightarrow 0, \end{aligned} \quad (4.37)$$

from (4.36), we have $\frac{|J(\mathbf{u}_k)|}{\|\mathbf{u}_k\|^2} \leq C$, so by dividing the left hand side of (4.37) with $\|\mathbf{u}_k\|^2$ there holds

$$\left| \int_{\mathbf{R}^N} \frac{\nabla_z W(x, \mathbf{u}_k(x)) \cdot \mathbf{u}_k(x)}{\|\mathbf{u}_k\|^2} dx \right| \leq C' \quad (4.38)$$

for some constant $C' > 0$. On the other hand, Since $\mathbf{v}(x) \neq 0$ in some positive measure set $\Omega \subset \mathbf{R}^N$, so $\mathbf{v}_k(x) \neq 0$ for large k , and $|\mathbf{u}_k(x)| \rightarrow +\infty$ as $k \rightarrow \infty$, for any fixed $x \in \Omega$.

So by (W₂), we have

$$\lim_{k \rightarrow \infty} |\mathbf{v}_k(x)|^2 \frac{2W(x, \mathbf{u}_k(x))}{|\mathbf{u}_k|^2} = +\infty, \quad \forall x \in \Omega. \quad (4.39)$$

By Remark (1) before Theorem 1.1, we have

$$\nabla_z W(x, \mathbf{u}_k(x)) \cdot \mathbf{u}_k(x) \geq 2W(x, \mathbf{u}_k(x)).$$

So as $k \rightarrow +\infty$, we have

$$\begin{aligned} & \int_{\mathbf{R}^N} \frac{\nabla_z W(x, \mathbf{u}_k(x)) \cdot \mathbf{u}_k(x)}{\|\mathbf{u}_k\|^2} dx = \int_{\{\mathbf{v}_k(x) \neq 0\}} |\mathbf{v}_k(x)|^2 \frac{\nabla_z W(x, \mathbf{u}_k(x)) \cdot \mathbf{u}_k(x)}{|\mathbf{u}_k(x)|^2} dx \\ & \geq \int_{\mathbf{R}^N} \chi_{\{\mathbf{v}_k \neq 0\}}(x) |\mathbf{v}_k(x)|^2 \frac{2W(x, \mathbf{u}_k(x))}{|\mathbf{u}_k(x)|^2} dx \geq \int_{\Omega} \chi_{\{\mathbf{v}_k \neq 0\}}(x) |\mathbf{v}_k(x)|^2 \frac{2W(x, \mathbf{u}_k(x))}{|\mathbf{u}_k(x)|^2} dx \rightarrow \infty, \end{aligned}$$

this contradicts to (4.38). There is another explanation about the above estimate. We observe that there exists $\delta > 0$ such that $meas\{x \in \Omega \mid |\mathbf{v}(x)| \geq \delta\} > 0$. Otherwise, $\forall n \in \mathbb{N}$, $meas\{x \in \Omega \mid |\mathbf{v}(x)| \geq \frac{1}{n}\} = 0$. Set $\Omega_n = \{x \in \Omega \mid |\mathbf{v}(x)| \geq \frac{1}{n}\}$, then in $\Omega \setminus \bigcup_{n=1}^{+\infty} \Omega_n$, there holds $\mathbf{v}(x) = 0$. But $\Omega \setminus \bigcup_{n=1}^{+\infty} \Omega_n$ and Ω have the same measure, it is impossible. We may assume $meas \Omega < +\infty$, by Egorov's theorem, there exists a positive measure subset Ω_0 of $\{x \in \Omega \mid |\mathbf{v}(x)| \geq \delta\}$ such that \mathbf{v}_k uniformly convergent to \mathbf{v} , so for $k \geq K$ with K large, there holds $|\mathbf{v}_k(x)| \geq \delta/2$ in Ω_0 . Thus (4.39) holds in Ω_0 . So there holds

$$\int_{\{\mathbf{v}_k(x) \neq 0\}} |\mathbf{v}_k(x)|^2 \frac{\nabla_z W(x, \mathbf{u}_k(x)) \cdot \mathbf{u}_k(x)}{|\mathbf{u}_k(x)|^2} dx \geq \int_{\Omega_0} |\mathbf{v}_k(x)|^2 \frac{2W(x, \mathbf{u}_k(x))}{|\mathbf{u}_k(x)|^2} dx \rightarrow \infty.$$

If $\mathbf{v} = 0$ in \mathcal{H} , inspired by [14], we choose $t_k \in [0, 1]$ such that $\Psi(t_k \mathbf{u}_k) := \max_{t \in [0, 1]} \Psi(t \mathbf{u}_k)$.

For any $\beta > 0$ and $\tilde{\mathbf{v}}_k := (4\beta)^{1/2} \mathbf{v}_k \rightharpoonup 0$, by Lemma 3.4 and the compactness of P' (see Lemma 1.22 in [18]) we have that $J(\tilde{\mathbf{v}}_k) \rightarrow 0$ and $\int_{\mathbf{R}^N} W(x, \tilde{\mathbf{v}}_k(x)) dx = P(\tilde{\mathbf{v}}_k) - P(0) = \langle P'(\xi_k \tilde{\mathbf{v}}_k), \tilde{\mathbf{v}}_k \rangle = \langle P'(\xi_k \tilde{\mathbf{v}}_k) - P'(0), \tilde{\mathbf{v}}_k \rangle + \langle P'(0), \tilde{\mathbf{v}}_k \rangle \rightarrow 0$ as $k \rightarrow \infty$, here $\xi_k \in (0, 1)$. So there holds

$$\Psi(t_k \mathbf{u}_k) \geq \Psi(\tilde{\mathbf{v}}_k) = 2\beta - \lambda J(\tilde{\mathbf{v}}_k) - \int_{\mathbf{R}^N} W(x, \tilde{\mathbf{v}}_k(x)) dx \geq \beta,$$

when k is large enough. By the arbitrariness of β , it implies that

$$\lim_{k \rightarrow \infty} \Psi(t_k \mathbf{u}_k) = \infty. \quad (4.40)$$

Since $\Psi(0) = 0$, $\Psi(\mathbf{u}_k) \rightarrow c$, we have $t_k \in (0, 1)$. By the definition of t_k ,

$$\langle \Psi'(t_k \mathbf{u}_k), t_k \mathbf{u}_k \rangle = 0. \quad (4.41)$$

From (4.40), (4.41), we have

$$\begin{aligned} & \Psi(t_k \mathbf{u}_k) - \frac{1}{2} \langle \Psi'(t_k \mathbf{u}_k), t_k \mathbf{u}_k \rangle \\ & = \int_{\mathbf{R}^N} \left(\frac{1}{2} \nabla_z W(x, t_k \mathbf{u}_k(x)) \cdot t_k \mathbf{u}_k(x) - W(x, t_k \mathbf{u}_k(x)) \right) dx \rightarrow \infty. \end{aligned} \quad (4.42)$$

By (W₄), there exists $\theta \geq 1$ such that

$$\begin{aligned} & \int_{\mathbf{R}^N} \left(\frac{1}{2} \nabla_z W(x, \mathbf{u}_k(x)) \cdot \mathbf{u}_k(x) - W(x, \mathbf{u}_k(x)) \right) dx \\ & \geq \frac{1}{\theta} \int_{\mathbf{R}^N} \left(\nabla_z W(x, t_k \mathbf{u}_k(x)) \cdot t_k \mathbf{u}_k(x) - W(x, t_k \mathbf{u}_k(x)) \right) dx, \end{aligned} \quad (4.43)$$

Hence

$$\int_{\mathbf{R}^N} \left(\frac{1}{2} \nabla_z W(x, \mathbf{u}_k(x)) \cdot \mathbf{u}_k(x) - W(x, \mathbf{u}_k(x)) \right) dx \rightarrow \infty. \quad (4.44)$$

On the other hand,

$$\int_{\mathbf{R}^N} \left(\frac{1}{2} \nabla_z W(x, \mathbf{u}_k(x)) \cdot \mathbf{u}_k(x) - W(x, \mathbf{u}_k(x)) \right) dx = \Psi(\mathbf{u}_k) - \frac{1}{2} \langle \Psi'(\mathbf{u}_k), \mathbf{u}_k \rangle \rightarrow c. \quad (4.45)$$

(4.44) and (4.45) are contradiction. Hence $\{\mathbf{u}_k\}$ is bounded in \mathcal{H} . So up to a subsequence, we can assume that $\mathbf{u}_k \rightharpoonup \mathbf{u}$ for some \mathcal{H} .

Since $\Psi'(\mathbf{u}_k) = E'(\mathbf{u}_k) - \lambda J'(\mathbf{u}_k) - P'(\mathbf{u}_k) \rightarrow 0$ and J', P' are compact, we have that $E'(\mathbf{u}_k) \rightarrow \lambda J'(\mathbf{u}) + P'(\mathbf{u})$ in \mathcal{H} . So

$$\langle E'(\mathbf{u}_k), \mathbf{u}_k - \mathbf{u} \rangle = \langle E'(\mathbf{u}_k) - (\lambda J'(\mathbf{u}) + P'(\mathbf{u})), \mathbf{u}_k - \mathbf{u} \rangle + \langle \lambda J'(\mathbf{u}) + P'(\mathbf{u}), \mathbf{u}_k - \mathbf{u} \rangle \rightarrow 0.$$

By Lemma 3.2, $\mathbf{u}_k \rightarrow \mathbf{u}$ in \mathcal{H} . ■

Remark 4.4 If we replace the space \mathcal{H} by \mathcal{H}_r , then Lemma 4.1, 4.2, 4.3 also hold.

Proof of Theorem 1.1 Define D_-, S_+, Q, H as Theorem 2.1, then from Lemma 4.1, $\Psi(\mathbf{u}) \geq \alpha > 0$ for every $\mathbf{u} \in S_+$, from Lemma 4.2, $\Psi(\mathbf{u}) \leq 0$ for every $\mathbf{u} \in D_- \cup H$ and Ψ is bounded on Q . Applying Lemma 4.3, it follows from Theorem 2.1 that Ψ has a critical value $d \geq \alpha > 0$. Hence \mathbf{u} is a non-trivial weak solution of (1.1).

For the cases $0 \leq \lambda < \mu_1$ or $V_1^+(x) \equiv 0 \equiv V_2^+(x)$, set $C_- = \{0\}$ and $C_+ = \mathcal{H}$, it is easy to see that the arguments above are also valid. The proof of Theorem 1.1 is complete. ■

Proof of Theorem 1.2 By Remarks 3.11 and 4.4, the proof is the same as that of Theorem 1.1, we only need to replace the space \mathcal{H} by \mathcal{H}_r . ■

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